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On the Perfect Groups.

By G. A. Miller.

The necessary and sufficient condition that a group is *solvable* is that its α^{th} derivative (derived group) is unity.[*] When no lower than the α^{th} derivative is unity, the $(\alpha - 1)^{\text{th}}$ derivative must be an Abelian group whose order exceeds unity. All the derivatives which are lower than the $(\alpha - 1)^{\text{th}}$ must then be non-Abelian. Dedekind calls the first derivative the commutator subgroup. The more general notation which we employ is due to Lie.

When a group is *insolvable* its α^{th} derivative must be a *perfect*[†] group whose order exceeds unity. The factors of composition of this perfect group include all the composite factors of composition of the original insolvable group. Hence the study of insolvable groups is reduced to that of perfect groups. In other words, an insolvable group is either perfect or it contains one and only one perfect group as a characteristic subgroup with respect to which its quotient group is solvable.

Since a perfect group is identical with its derivatives, it cannot be isomorphic to any Abelian group whose order exceeds unity. Conversely, if a group is not isomorphic to any Abelian group whose order exceeds unity, it must coincide with its first derivate, and hence it must be perfect. The totality of perfect groups, therefore, includes that of simple groups of a composite order, but it is included in that of insolvable groups.

It is easy to see that no two of these three totalities are identical, for the direct product of any number of simple groups of composite orders is evidently a compound perfect group, while the direct product of a perfect and a solvable group is insolvable without being perfect. All the symmetric groups whose orders exceed 24 are also insolvable without being perfect. We proceed to consider some general properties of any perfect group.

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[*] Quarterly Journal of Mathematics, vol. 28, p. 268; cf. Frobenius, Sitzungsberichte der Berliner Akademie, 1896, p. 1348.

[†] A perfect group is identical with its first derivative (Lie).

Theorem I.—Every perfect group has an α, 1 *isomorphism to a simple group of composite order.*

When $\alpha = 1$ the perfect group is simple and of a composite order, and vice versa. When the perfect group is compound it must have an i, 1 isomorphism to a *perfect* group of lower order, for if this group of lower order were imperfect the original group would have to be imperfect. We repeat this process if the given perfect group of lower order is not simple. We shall thus finally arrive at a simple group of composite order to which the original perfect group has an α, 1 isomorphism. The order of this simple group is a factor of composition of the original group.

Corollary I.—If an imprimitive substitution group is perfect it must permute all its systems of imprimitivity according to a perfect group.

Corollary II.—If a perfect group has prime factors of composition it must contain a solvable characteristic subgroup whose order is the product of all these prime factors.

Theorem II.—If a perfect group is represented as an intransitive substitution group, all its transitive constituents must be perfect.

If a transitive constituent were imperfect the first derivative of the group could not include all the substitutions of this constituent. It could therefore not contain all the substitutions of the group which is supposed to be perfect. This is impossible.

Corollary.—A transitive constituent of a perfect substitution group contains only positive substitutions.

Theorem III.—If a transitive substitution group of a prime degree is perfect it is also simple.

If such a group were compound it would contain a transitive invariant (self-conjugate) subgroup whose order would be less than the order of the group. This subgroup would contain all the cyclical substitutions of the prime degree (p) that are contained in the entire group. With respect to this subgroup the entire group would have an α, 1 isomorphism to a cyclical group of order $(p - 1) \div \beta$, β being an integer. As a perfect group cannot have such an isomorphism, the given group cannot be compound.

Corollary.—A transitive group of a prime degree cannot contain more than one composite factor of composition.

If it contained two such factors both of them would be factors of composition of its perfect characteristic subgroup with respect to which it is isomorphic to a solvable group. This is impossible, since this characteristic subgroup is transitive.

Theorem IV.—If a transitive substitution group of degree $2p$, p being any prime number, is perfect, it is either simple or imprimitive. In the latter case it must contain p systems of imprimitivity, and it must permute them according to a simple group of degree p.

Since all the groups whose degrees are less than five are solvable, we may assume that p is odd. If the required perfect group were primitive and compound, all its substitutions of order p would be contained in an invariant transitive subgroup whose order would be less than the order of the group.* With respect to this subgroup the perfect group would have to be isomorphic to an Abelian group. As this is impossible, the given perfect group is simple if it is primitive. If the required perfect group is imprimitive it must permute its systems of imprimitivity according to a perfect transitive group of degree p. We proved above that such a group is simple.

Theorem V.—Any simple quotient group of a compound substitution group of degree n may be represented as a transitive group whose degree is less than n.

If the given group (G) of degree n is intransitive, it must contain at least one transitive constituent which has an α, 1 (α being an integer) isomorphism to the given quotient group, since the latter is simple. Hence we may confine our attention to the case when G is transitive, and we may suppose that n is the smallest number of elements by means of which G can be represented transitively.

If a subgroup of G, which contains all its substitutions that do not contain a given element, corresponds to only a part of the operators of the simple quotient group (S), it is possible to represent S as a transitive group of degree $n \div m$, $m > 1$. If this subgroup is simply isomorphic to S, it is possible to represent S as a transitive group whose degree cannot exceed $n - 1$. If this subgroup is multiply isomorphic to S, we may use it or one of its transitive constituents in place of G. In either case the new transitive group of which S is also a quotient group is of lower order and of lower degree than G. As this process cannot be repeated indefinitely the theorem is proved.

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* Jordan, Bulletin de la Société Mathématique de France, t. 1, p. 41.

Corollary.—If a quotient group of G is perfect and compound, G is isomorphic to some simple group of composite order which can be represented by a smaller number of elements than is required to represent G.

By means of these theorems we may readily determine a large number of perfect groups. As all the simple groups of composite order are perfect we need not consider these. We proceed to seek all the compound perfect groups which may be represented as substitution groups with 11 or a smaller number of elements. According to the preceding theorems the lowest possible degree of such a group is 8, and if a compound group of this degree is perfect it must be primitive. As 8 is equal to a prime number plus 3, the order of a primitive group of this degree is not divisible by 5 unless it is either the alternating or the symmetric group.* The composite factor of composition of the required group must therefore be 168 and its order cannot be less than $168 \times 8 = 1344$, since it must contain a transitive invariant subgroup. There is only one primitive group of this degree that satisfies the last condition and does not contain the alternating group, viz. the well-known triply transitive group of order 1344.† That this group is perfect follows directly from the facts that it contains no invariant subgroup of order 168, and the seven operators of order 2 in its invariant subgroup of order 8 are conjugate. Hence *there is one and only one compound perfect group of degree* 8, *and there is no such group for any lower degree.*

If a group of degree 9 were compound and perfect it would also be primitive according to the given theorems. As its order could not be divisible by 5, and as it could not contain two composite factors of composition, it would have to be isomorphic to the simple group of order 168 with respect to a transitive invariant subgroup. It could not be isomorphic to the simple group of order 504, since this cannot be represented by less than 9 elements. Hence the order of the required group could not be less than $168.9 = 1512$. But the primitive group of this order does not contain an invariant subgroup of order 9, and there is no larger primitive group of degree 9 that does not include the alternating group of this degree. Hence *there is no compound perfect group of degree* 9.

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*Jordan, loc. cit.

†This group is given by Kirkman, Proceedings of the Literary and Philosophical Society of Manchester (1863), vol. 3; by Jordan, Comptes Rendus, vol. 73; by Noether, Mathematische Annalen, vol. 15; and by others. It is singular that Wiman supposed that he established its existence for the first time, Nachrichten, Göttingen (1897), p. 58.

Since a perfect group of a prime degree must be simple, and there is only one simple group of degree 5 and all the groups of a lower degree are solvable, there can be only one compound perfect intransitive group of degree 10. It is of order 3600, being the direct product of two simple groups of order 60. An imprimitive compound perfect group of this degree must contain 5 systems of imprimitivity, and it must permute them according to the alternating group of degree 5. Its head can contain only positive substitutions and its order (2^α) must satisfy the congruence

$$2^\alpha \equiv 1 \text{ mod. } 5 \qquad 1 < \alpha < 5.$$

Hence $\alpha = 4$, and the order of the imprimitive group is $16.60 = 960$. There is only one imprimitive group of this degree and order.* If it were not perfect it would contain an invariant subgroup of order 60, since no subgroup, whose order exceeds unity, that is contained in the head can be invariant. Each substitution of an invariant subgroup of order 60 would have to be commutative with every substitution of the head. As this condition could not be satisfied by the substitution of order 5, the given imprimitive group must be a compound perfect group.

If a primitive group of degree 10 were compound and perfect it would be isomorphic to a simple group of a lower degree whose order is not divisible by 7, since a cyclical substitution of a prime degree (p) cannot occur in any primitive groups except those of degrees p, $p+1$, $p+2$, and those of higher degrees which include the alternating group of their own degree. As a transitive group would correspond to identity in the isomorphic simple group, and the order of such a simple group would be divisible by 5, the order of the required primitive group would be divisible by 25. This is clearly impossible, since a group of degree 10 and order 25 must contain a cyclical substitution of order 5. Hence *there are two and only two compound perfect groups of degree* 10, *the one is intransitive and the other is imprimitive.*

The compound perfect groups of degree 11 must be intransitive and their transitive constituents must be simple and of degrees 5 and 6. Hence there are two such groups, viz. the direct product of the alternating groups of these degrees and the direct product of the alternating group of degree 5 and the primitive group of degree 6 which is simply isomorphic to it. Their orders are 21600 and

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* Cole, Quarterly Journal of Mathematics, vol. 27, p. 42.

3600 respectively. We give below the enumerations of the simple, the perfect, and the insolvable groups which may be represented by 11 or a smaller number of elements. The lowest order of a compound perfect group is 120. There is only one such group of this order. As a substitution group it can be represented by 24, but by no smaller number of elements.

Degree……………………	5	6	7	8	9	10	11
Number of simple groups*….	2	2	3	2	2	4	5
Number of perfect groups….	1	2	2	3	2	6	6
Number of insolvable groups..	2	4	6	16	41	106	228

CHICAGO, *Dec.* 1897.

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* This enumeration includes all the possible substitution groups of the given degrees. If we regard these groups as operation groups they are not all distinct.